TOPIC: Binomial Distribution

Before we get to the binomial distribution, let's do a quick review of permutations and combinations.

Permutations:

A permutation is a particular arrangement.

Example: How many ways can you arrange the letters A, B, and C?

ABC
ACB
BAC
BCA
CAB
CBA

How did we get this? Let's use 3 imaginary slots:

Slot #1 Can be:		Slot #2 Can be:	Slot #3 Can only be:	Result:
A	→	B	C	ABC
	→	C	B	ACB
B	→	A	C	BAC
	→	C	A	BCA
C	→	A	B	CAB
	→	B	A	CBA

On your scientific calculator, you will use the **nPr** key. The P stands for permutations. n is the number of distinct objects you wish to arrange and r is the number of slots or spaces. Thus, in the above example, we have 3 objects to arrange in three slots, or, 3**P**3 = 6 .

The general formula is: $_nP_r = \frac{n!}{(n-r)!}$

[Note: 0! ≡ 1]

When n = r, nPn = n!

n! is read as *n factorial*. In general,

$$n! = n \cdot (n-1) \cdot (n-2) \cdot \ldots \cdot (2) \cdot (1) = n \cdot (n-1)!$$

So,

10! = 10 x 9 x 8 x 7 x 6 x 5 x 4 x 3 x 2 x 1 = 3,628,800

6! = 6 x 5 x 4 x 3 x 2 x 1 = 720

Example: How many ways can you assign 5 workers to 5 different tasks?

Ans: 5P5 = 5! = 120

Example: How many ways can you arrange 10 books if you have space for 5 books in your bookcase?

Ans.: 10P5 = 10! / 5! = 10 x 9 x 8 x 7 x 6 = 30,240 (Use your calculator!)

Example: a wedding planner needs to seat 10 people at a round table that has place for 10. How many different arrangements can she make?

Ans: $_{10}P_{10} = 10! = 3{,}628{,}800$

[Follow up question: how many different potential family feuds can break out?]

Combinations:

With permutations, the *arrangement* of the items is important. Each unique sequence is another permutation. Thus, ABC was different from BCA and both were different from CBA.

With combinations, however, ABC, BCA, and CBA are all the same. They are all part of the same combination.

Example: How many different groups of 3 can be selected from 7 people? Say these people are named A, B, C, D, E, F, G. Note that once you select, say, B, D, and E, the six different arrangements you can make from them are irrelevant.

The formula for combinations is:

$$nCr = \frac{n!}{r!(n-r)!}!$$

Note that $nCr = \frac{n\,\mathrm{Pr}}{r!}$

You can use the nCr key on your calculator.

Ans: 7**C**3 = $\frac{7!}{3!4!}$ = 35

Example: How many different hands can one draw in a game of 7-card rummy?

Ans: $_{52}C_7 = \frac{52!}{7!\,45!}$ = 133,784,560

Example: How many samples of size n=6 can be drawn from a population of size N=50?

Ans: $_{50}C_6 = \frac{50!}{6!\,44!}$ = 15,890,700

Example: How many ways can one come up with 2 heads in 3 tosses of a coin?

Ans: $3\mathbf{C}2 = \frac{3!}{2!\,1!} = 3$

Let's see why this is true. When a coin is tossed 3 times there are 8 possible outcomes. Let's enumerate them:

HHH
HHT
HTH
HTT
THH
THT
TTH
TTT

Note that only **3** of these possible outcomes have exactly 2 heads (and 1 tail).

The Binomial Probability Distribution

Example: An air conditioner made by a certain company is made of 20 distinct parts. Each part has a .004 probability of being defective. What is the probability that a randomly selected air conditioner will not work perfectly? [Note that it will not work perfectly if even one of these parts is defective.]

The Binomial Distribution is used when the sampling process works this way:

a) there are only two possible outcomes for each trial, object, or observation. These two outcomes are mutually exclusive. These outcomes are called *success* and *failure*.

Examples: heads / tails; pass / fail; defective / non-defective; dead / alive; hit / miss

b) The outcomes are independent events.

c) The probability of success (or failure) is constant from trial to trial. For example, the probability of getting a head on a coin toss is the same at every toss of the coin.

[This sampling process is called a *Bernoulli process*.]

The binomial distribution can be used to compute the probability of getting a particular number of successes in a specified number of trials (or, observations).

p	≡	the probability of success
(1 – p)	≡	the probability of failure
x	≡	# successes
(n – x)	≡	# failures
n	≡	# observations

The probability of getting x successes out of n trials (or observations) is:

$$P(x) = nCx\ p^{x}\ (1\text{-}p)^{n-x}$$

Let's try this with the previous problem:
What is the probability of getting 2 Heads in 3 tosses of a fair coin? Should be 3 out of 8, right? Let's see:

$P(2) = 3C2 \left(\frac{1}{2}\right)^{2} \left(\frac{1}{2}\right)^{1}$

$= 3\left(\frac{1}{2}\right)^{3}$

$= 3\left(\frac{1}{8}\right)$

$= \left(\frac{3}{8}\right)$

YES!

So…

The probability of getting x successes out of n trials (or observations) is:

$$P(x) = nCx\ p^{x}\ (1\text{-}p)^{n\text{-}x}$$

The mean (expected value) of a binomial distribution is:

$$\mu = np$$

The variance of a binomial distribution is:

$$\sigma^2 = np(1–p)$$

Example: if you toss a coin 10 times, what is the probability of getting exactly 5 heads?

Ans: p=.5 x=5 n=10

$_{10}C_5 (.5)^5(.5)^5 = .2461$

Note that the mean is 5 heads. On average, when you toss a coin 10 times, you expect to get 5 heads. This is the most likely outcome, 5 heads.

> [If you toss a coin 100 times, you expect to get 50 heads (the mean). That is the most likely outcome. However, the probability of getting exactly 50 heads in 100 tosses is less than 8% (.0796). What you actually expect is getting 50 heads ± 10 heads (the variance is 25 which means the standard deviation is 5 heads) or 40 to 60 heads. You will get 40 to 60 heads about 95.5 % of the time. We are going to learn about confidence intervals in the second half of the course so you may not understand this now.]

Example: A machine produces parts that are very difficult to make. It turns out that 1 out of 20 are defective and must be thrown out. What is the probability that a sample of 10 parts will contain 0 defectives?
Note: If 1 out of 20 parts are defective, this means that p = .05.

Ans: $10\mathbf{C}0(.05)^{0}(.95)^{10} = .5987$

Example: 60% of the students at CUNY are female. What is the probability that a randomly selected group of 25, there will be exactly 15 females?

Ans: $25\mathbf{C}15(.60)^{15}(.40)^{10} = .1612$

Example: Suppose the probability that a person who is 50 years old will die before s/he reaches the age of 55 is .001. What is the likelihood that in a group of one thousand 50-year-olds, exactly 2 will die before reaching the age of 55?

Ans: $1000\mathbf{C}2(.001)^{2}(.999)^{998} = .1840$

Example: An air conditioner made by a certain company is made of 20 distinct parts. Each part has a .004 probability of being defective. What is the probability that a randomly selected air conditioner will not work perfectly?

Ans: Best way to solve this problem is to answer the question of what is the probability that the a/c will work perfectly, i.e., has 0 defectives.

Probability that the a/c will work (0 defectives) is: $_{20}C_0(.004)^0(.996)^{20} = .923$
Probability that it will not work (i.e., has 1 or more defective parts) is 1 - .923 = .077. There is a 7.7% chance that a randomly selected a/c made by this company will not work. What should be done to improve quality at this company? One solution is to reduce the number of parts. Also, improve quality of each part. .004 defective rate is not very good. You will learn about 6 sigma quality in other courses.

Example: A hybrid car made by a certain company consists of 400 distinct components. Each component has a .001 chance of being defective. What is the probability that a car made by this company will not work perfectly?

Ans: Probability that the car will work perfectly (0 defectives) is:
$400\mathbf{C}0(.001)^{0}(.999)^{400} = .6702$
Probability that it will not work perfectly is 1 - .6702 = .3298 -- There is almost a 33% chance that something will be wrong with the car.