TOPIC: PROBABILITY

The word probability is actually undefined, but the <u>probability of an event</u> can be explained as the proportion of times, under identical circumstances, that the event can be expected to occur. It is the event's long-run frequency of occurrence.

For example, the probability of getting a head = .5. If you keep tossing the coin, you will note that a head occurs about one half of the time.

<u>Objective probabilities</u> are long-run frequencies of occurrence, as above. Probability, in its classical (or, objective) meaning refers to a repetitive process, one which generates outcomes which are not identical and not individually predictable with certainty but which may be described in terms of relative frequencies. These processes are called <u>stochastic processes</u> (or, chance processes). The individual results of these processes are called <u>events</u>.

<u>Subjective probabilities</u> measure the strengths of personal beliefs. For example, the probability that a new product will succeed.

<u>Random variable</u>: That which is observed as the result of a stochastic process. A random variable takes on (usually numerical) values. Associated with each value is a probability that the value will occur.

For example, when you toss a die:
$P(1) = P(2) = P(3) = P(4) = P(5) = P(6) = 1/6$

<u>Simple probability</u>: P(A). The probability that an event (say, A) will occur.

<u>Joint probability</u>: P(A and B). $P(A \cap B)$. The probability of events A and B occurring together.

<u>Conditional probability</u>: P(A|B), read "the probability of A given B." The probability that event A will occur given event B has occurred.

<u>Probability vs. Statistics</u>: In probability you know the population and in statistics you want to draw inferences about the population from the sample.

Rules of Probability

1. $0 \leq P(A) \leq 1$

The probability of an event (say, event A) cannot be less than zero or greater than one.

2. $\sum P_i = 1$ or $P(A) + P(A') = 1$

The sum of the probabilities of all possible outcomes (events) of a process (or, experiment) must equal one. [A' means *not A*.]
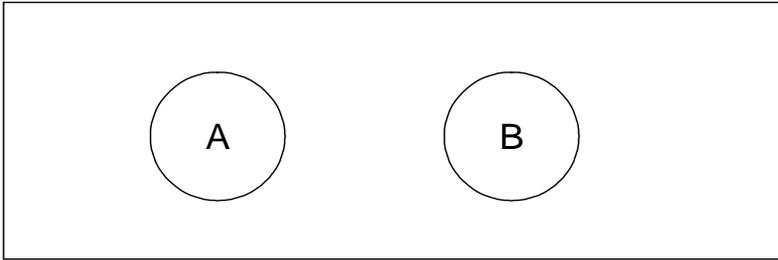
3. Rules of addition.

    a. $P(A \text{ or } B) = P(A \cup B)$
 $= P(A) + P(B)$ **if** events A and B are mutually exclusive.

    Two events are <u>mutually exclusive</u> if they cannot occur together. E.g., male or female; heads or tails.

    b. $P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$

    This is the <u>general formula</u> for addition of probabilities, for any two events A and B.   P(A and B) is the joint probability of A and B occurring together and is equal to zero if they are mutually exclusive (i.e., if they cannot occur together).

Venn Diagram: Two mutually exclusive events.
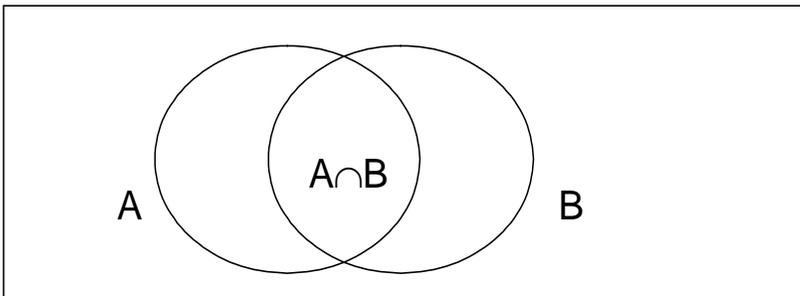
Venn Diagram: A and B are not mutually exclusive.

P(A∩B) is the intersection of A and B Therefore the general formula for addition is

P(A or B) = P(A∪B) = P(A) + P(B) – P(A and B)

Also,   P(A′) = P(not A) = 1 – P(A)
          P(A′ or B′) = 1 – P(A and B)

4.   Rules of multiplication are used for determining joint probabilities, the probability that events A and B will occur together.

a.   P(A and B) = P(A ∩ B)
                                = P(A) P(B) **if** events A and B are independent.
Events A and B are <u>independent</u> if knowledge of the occurrence of B has no effect on the probability that A will occur.

For example, P(Blue eyes | Male) = P(Blue eyes) because eye color and sex are independent.
However,
       P(over 6 feet tall | Female) ≠ P(over 6 feet tall)
       P(getting into car accident | under 26) ≠ P(getting into car accident)

b.   P(A and B) = P(A|B) P(B) = P(B|A) P(A)
This is the general formula for multiplying probabilities, for any two events A and B not necessarily independent.

P(A|B) is a conditional probability. If events A and B are independent, then P(A|B) = P(A), and P(A and B) becomes equal to P(A)P(B), as above.

A and B are independent if:
       P(A|B) = P(A) or if
       P(A and B) = P(A)P(B)

5.   Conditional probability. From the formula in 4.b., we see that we can compute the conditional probability as
                         P(A|B) = P(A and B) / P(B)

6.   Bayes' Theorem [optional]
       Since   P(A|B) = P(A and B) / P(B)
       and   P(A and B) = P(B|A) P(A)
       therefore
                         P(A|B) = P(B|A) P(A) / P(B)

 [We generally need to compute P(B)].

Example. Toss of a Die

$P(1) = 1/6$
$P(2) = 1/6$
$P(3) = 1/6$ {mutually exclusive}
$P(4) = 1/6$
$P(5) = 1/6$
$P(6) = 1/6$

$\sum P_i = 1$

$P(1 \text{ or } 3) = P(1) + P(3) - P(1 \text{ and } 3) = 1/6 + 1/6 - 0 = 1/3.$

When rolling a die once, what is the probability that:
(a) the face of the die is odd?
(b) the face is even or odd?
(c) the face is even or a one?
(d) the face is odd or a one?
(e) the face is both even and a one?
(f) given the face is odd, it is a one?

Example. Reading *New York Times* (T) and *Wall Street Journal* (W).

In a small village in upstate New York, Given:
P (Reading T) =   .25
P (Reading W) =   .20
P (T and W) =   .05

Question:
What is the probability of being either a *NYT* or *WSJ* reader?

Answer:
P(T or W)   = P(T) + P(W) – P (T and W)
             = .25 + .20 - .05
             = .40

Another way to solve this problem, by Venn Diagram:
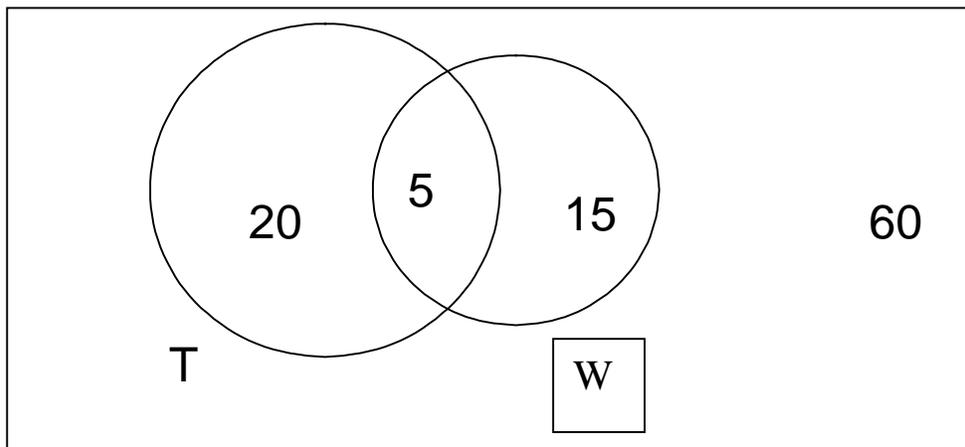
So, out of 100 people in total:
25 people read *NYT* (T);
20 people read *WSJ* (W);
5 people read both;
60 people read neither.

Thus, it can be easily seen that 40 people (out of 100) read either *NYT* or *WSJ*.

Other Probabilities:
P(T′ and W′) = .60
P(T′ and W) = .15
P(T and W′) = .20
P(T or W) = 1 – P(T` and W`) = 1– .60 = .40

P (T′ or W′) = P(T′) + P(W′) – P(T′ and W′)
              =.75 + .80 - .60
              = .95

Note that:
P(T′ or W′) = 1- P(T and W) = 1 - .05 = .95

Another way to solve this problem is to construct a joint probability table:

Joint Probability Table

	T	T′		
W	.05	.15		.20
W′	.20	.60		.80
	.25	.75		1.00

Important: Do not confuse mutually exclusive and independent. Mutually exclusive means that two things cannot occur at the same time [P (A and B) = 0]. You cannot get a head and tail at the same time; you cannot be dead and alive at the same time; you cannot pass and fail a course at the same time; etc.

Independence has to do with the effect of, say B, on A. If knowing about B has no effect on A, then they are independent. It is very much like saying that A and B are unrelated. Are waist size and gender independent of each other. Suppose I know that someone who is an adult and has a 24-inch waist, does that give me a hint as to whether they are male or female? How many adult men have a 24-inch waist? How many women? Is the probability (24-inch waist/adult male) = probability (24 inch waist/adult female).   I suspect that the probabilities are not the same. There is a relationship between gender and waist size (also hand size and height for that matter). Thus, they are not independent.

Researchers are always testing for relationships. Sometimes we want to know whether two variables are related. Is there a relationship between cigarette smoking and cancer (see next problem) or are they independent? We know the answer to that one.

Is there a relationship between your occupation and how long you will live (longevity) or are they independent? Studies show that they are not independent. Librarians and professors have relatively long life spans (perhaps not the ones that teach online courses ☺); coal miners have the shortest life spans. Drug dealers (is that an occupation?) also have very short life spans.

Is there a relationship between obesity and diabetes? Studies show that these two factors are not independent.

Example: Smoking and Cancer

Contingency Table

	S (smoker)	S′ (non-smoker)	
C (cancer)	100	50	150
C′ (no cancer)	300	550	850
	400	600	1000

Are Smoking and Cancer independent?

Joint Probabilities
P(C and S) = .10
P(C` and S) = .30
P(C and S`) = .05
P(C` and S`)= .55

Marginal Probabilities
P(C) = .15   (150/1000)
P(C`) = .85   (850/1000)
P(S) = .40   (400/1000)
P(S`) = .60   (600/1000)

Question: Are smoking and cancer independent?

Answer check: Are the following probabilities equal or not?

   P(C) ?= P(C|S) ?= P(C|S`)

P(C|S) = P(C and S)/P(S)   = .10/.40 = .25
P(C|S′) = P(C and S′)/P(S′) = .05/.60 = .083
P(C)   = .15

Thus, cancer and smoking are not independent. There is a relationship between cancer and smoking.

Note: P(C) is a weighted average of P(C|S) and P(C|S′)
i.e.,   P(C) = P(S)P(C|S) + P(S′)P(C|S′) = .40(.25) + .60 (.0833) = .15

Alternate method:
If C and S are independent, then
P(C and S)   = P(C) P(S)
.10   ?= (.15)(.40)
.10 $\neq$ .06

This may be easier to do if we first construct a table of joint probabilities:

	S (smoker)	S′ (non-smoker)	
C (cancer)	100/1000	50/1000	150/1000
C′ (no cancer)	300/1000	550/1000	850/1000
	400/1000	600/1000	1000/1000

$\downarrow$

	S (smoker)	S′ (non-smoker)	
C (cancer)	.10	.05	.15
C′ (no cancer)	.30	.55	.85
	.40	.60	1.00

Notice that the marginal probabilities are the row and column totals. This is not an accident. The marginal probabilities are *totals* of the joint probabilities and weighted averages of the conditional probabilities.

Once we compute the table of joint probabilities, we can answer any question about a probability in this problem. The joint probabilities are in the center of the table; the marginal probabilities are in the margins, and the conditional probabilities are easily computed by dividing a joint probability by a marginal probability.

Example: Gender and Beer-Drinking

Contingency Table

	M (male)	F (female)		
B (beer drinker)	450	350		800
B′ (not a beer drinker)	450	750		1200
	900	1100		2000

⬇

	M (male)	F (female)		
B (beer drinker)	450/2000	350/2000		800/2000
B′ (not a beer drinker)	450/2000	750/2000		1200/2000
	900/2000	1100/2000		2000/2000

⬇

Table of Joint Probabilities

	M (male)	F (female)		
B (beer drinker)	.225	.175		.40
B′ (not a beer drinker)	.225	.375		.60
	.45	.55		1.00

<u>Joint Probabilities</u>

P(B and M) = .225

P(B and F)   = .175

P(B′ and M) = .225

P(B′ and F) = .375

<u>Marginal Probabilities</u>

P(B) = .40

P(B′) = .60

P(M) = .45

P(F)   = .55

Given that an individual is Male, what is the probability that that person is a Beer drinker?

   $P(B|M) = P(B \text{ and } M)/P(M) = .225/.45 = .50$

Given that an individual is Female, what is the probability that that person is a Beer drinker?

   $P(B|F) = P(B \text{ and } F)/P(F) = .175/.55 = .318$

Are beer drinking and gender independent?

Since,

   $P(B)   = .40$

Therefore, beer-drinking and sex are not independent.

Example: Marital Status and Microwave Ownership

	S (single)	S′ (not single)		
M (own microwave)	40	360		400
M′ (no microwave)	60	540		600
	100	900		1000

<u>Joint Probability</u>
P(S and M)   = .04
P(S′ and M) = .36
P(S and M`) = .06
P(S′ and M`) = .54

<u>Marginal Totals</u>
P(S) = .10
P(S′) = .90
P(M) = .40
P(M′) = .60

P(M|S) = .04/.10 = .40

P(M|S′) = .36/.90 = .40

P(M)   = .40

Therefore, microwave ownership and being single are independent.

Example: Marital Status and Depression

	S (single)	S′ (not single)		
D (Depressed)	800	200		1000
D′ (Not depressed)	3200	5800		9000
	4000	6000		10,000

<u>Joint Probability</u>
P(S and D)   = .08
P(S′ and D) = .02
P(S and D′) = .32
P(S′ and D′) = .58

<u>Marginal Totals</u>
P(S) = .40
P(S′) = .60
P(D) = .10
P(D′) = .90

P(D)   = .10

P(D|S) = .08/.40 = .20

P(D|S`) = .02/.60 = .033

Therefore, Depression and marital status are not independent.

Question: Does this prove that being single as an adult causes depression?

Example: Gender and Using Dove Soap

[from a random sample of 1,000 adults]

	M (male)	F (female)		
D (use Dove soap)	80	120		200
D′ (does not use Dove soap)	320	480		800
	400	600		1,000

Joint Probability

P(D and M) = .08
P(D and F)   = .12
P(D and M)   = .32
P(D′ and F) = .48

Marginal Totals

P(D) = .20
P(D′) = .80
P(M) = .40
P(F)   = .60

Are the events "Male" and "uses Dove soap" independent?

$$P(D) = 200/1000 = .20$$

$$P(D|M) = 800/1000 = .08/.40 = .20$$

$$P(D|F) = .12/.60 = .20$$

Yes, these two events are independent.

Alternative method:
P(M and D) = .08
P(M) = 400/1000 = .40
P(D) = 200/1000 = .20
.08 ?= (.40)(.20) Yes, they are independent.

Question:
Are gender and Dove soap usage independent?

Table of Joint Probabilities

	M (male)	F (female)		
D (use Dove soap)	.08	.12		20
D′ (does not use Dove soap)	.32	.48		80
	.40	.60		1.00