The Student's t Distribution

What do we do if (a) we don't know σ and (b) n is small? If the population of interest is normally distributed, we can use the Student's t-distribution in place of the standard normal distribution.

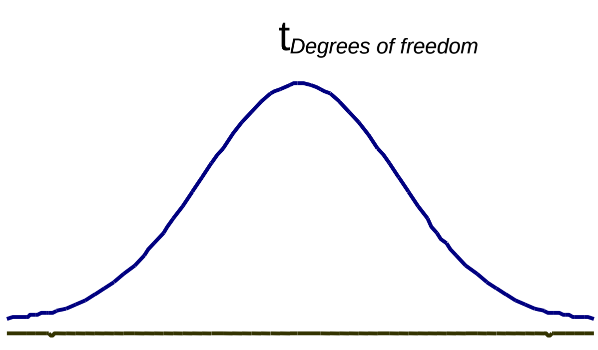
William S. Gossett who developed the t-distribution, wrote under the name "Student" since, as an employee of the Guiness Brewery in Dublin, he was required by the firm to use a pseudonym in publishing his results.

We will use Z for EITHER
(1) known σ OR (2) large n

Use t for
(1) Small sample AND
(2) Taken from N.D. population* AND
(3) Unkown σ

* What can we do if the population is not normally distributed and we have a small sample? Always use non-parametric statistical methods in this case.

The t-distribution looks like the normal distribution, except that it is more spread out. It is still symmetrical about the mean; mean=median=mode; goes from -∞ to +∞.



Student's t distribution is not a single distribution as is the standardized normal distribution (Z), but rather it is a series of distributions, one for each number of degrees of freedom. As n gets larger, student's t distribution approaches the normal distribution.

t statistic for testing hypotheses (n-1 degrees of freedom):

$$t_{n-1} = \frac{\overline{X} - \mu_0}{s_{\overline{X}}} \quad = \frac{\overline{X} - \mu_0}{\frac{s}{\sqrt{n}}}$$

$(1-\alpha)\%$ Confidence Interval Estimator

$$\overline{X} \pm t_{\alpha} \frac{s}{\sqrt{n}}$$

<u>Degrees of Freedom:</u>

# of degrees of freedom = the sample size minus 1. We "lose" a degree of freedom each time a statistic computed from the sample is used as a point estimator of a parameter. In this case, we do not know the population σ, so instead of:

$$\sigma = \sqrt{\frac{\sum (X - \mu)}{N}}$$

we use s to estimate σ:

$$s = \sqrt{\frac{\sum (X - \overline{X})}{n - 1}}$$

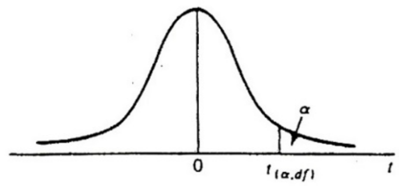
[s is an *unbiased* estimator of σ.]

We use $\overline{X}$ - a statistic - instead of μ in the formula for s. The price we pay is a loss of a degree of freedom. Notice that this penalty increases the size of the standard deviation.

Simple example: If $\overline{X}$=3 and when we have 5 numbers. 1, 2, 3, 4, __. The 5^{th} number must be 5 since $\overline{X}$=3 (which means the total must be 15). You only have leeway with 4 numbers — loss of 1 d.f.

The Short t Table: Critical Values of t

For a particular number of degrees of freedom, each entry represents the critical value of **t** corresponding to a specified upper tail area α.



Degrees of Freedom	*Upper Tail Areas* .25	.10	.05	.025	.01	.005
1	1.0000	3.0777	6.3138	12.7062	31.8207	63.6574
2	0.8165	1.8856	2.9200	4.3027	6.9646	9.9248
3	0.7649	1.6377	2.3 534	3.1824	4.5407	5.8409
4	0.7407	1.5332	2.1318	2.7764	3.7469	4.6041
5	0.7267	1.4 759	2.0150	2.5706	3.3649	4.0322
6	0.7176	1.4398	1.9432	2.4469	3.1427	3.7074
7	0.7111	1.4149	1.8946	2.3646	2.9980	3.4995
8	0.7064	1.3968	1.8595	2.3060	2.8965	3.3554
9	0.7027	1.3830	1.8331	2.2622	2.8214	3.2498
10	0.6998	1.3722	1.8125	2.2281	2.7638	3.1693
11	0.6974	1.3634	1.7959	2.2010	2.7181	3.1058
12	0;6955	1.3562	1. 7823	2.1788	2.6810	3.0545
13	0.6938	1.3502	1.7709	2.1604	2.6503	3.0123
14	0.6924	1.3450	1.7613	2.1448	2.6245	2.9768
15	0.6912	1.3406	1.7531	2.1315	2.6025	2.9467
16	0.6901	1.3368	1.7459	2.1199	2.5835	2.9208
17	0.6892	1.3334	1.7396	2.1098	2.5669	2.8982
18	0.6884	1.3304	1.7341	2.1009	2.5524	2.8784
19	0.6876	1.3277	1.7291	2.0930	2.5395	2.8609
20	0.6870	1.3253	1.7247	2.0860	2.5280	2.8453
21	0.6864	1.3232	1.7207	2.0796	2.5177	2.8314
22	0.6858	1.3212	1.7171	2.0739	2.5083	2.8188
23	0.6853	1.3195	1.7139	2.0687	2.4999	2.8073
24	0.6848	1.3178	1.7109	2.0639	2.4922	2.7969
25	0.6844	1.3163	1.7081	2.0595	2.4851	2.7874
26	0.6840	1.31 50	1.7056	2.0555	2.4786	2.7787
27	0.6837	1.3137	1.7033	2.0518	2.4727	2.7707
28	0.6834	1.3125	1.7011	2.0484	2.4671	2.7633
29	0.6830	1.3114	1.6991	2.0452	2.4620	2.7564
30	0.6828	1.3104	1.6973	2.0423	2.4573	2.7500
31	0.6825	1.3095	1.6955	2.0395	2.4528	2.7440
32	0.6822	1.3086	1.6939	2.0369	2.4487	2.1385
33	0.6820	1.3077	1.6924	2.0345	2.4448	2.7333
34	0.6818	1.3070	1.6909	2.0322	2.4411	2.7284
35	0.6816.	1.3062	1.6896	2.0301	2.4377	2.7238
36	0.6814	1.3055	1.6883	2.0281	2.4345	2.7195
37	0.6812	1.3049	1.6871	2.0262	2.4314	2.7154
38	0.6810	1.3042	1.6860	2.0244	2.4286	2.7116
39	0.6808	1.3036	1.6849	2.0227	2.4258	2.7079
40	0.6807	1.3031	1.6839	2.0211	2.4233	2.7045
41	0.6805	1.3025	1.6829	2.0195	2.4208	2.7012
42	0.6804	1.3020	1.6820	2.0181	2.4185	2.6981
4J	0.6802	1.3016	1.6811	2.0167	2.4163	2.6951
…	…	…	…	…	…	…
∞			**1.645**	**1.96**	**2.33**	**2.575**

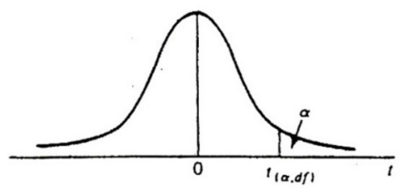
EXAMPLE:
In a sample of 16 management consultants, the following statistics relating to hourly wages have been computed:

$\overline{X}$ = $200

s = $96

(a) Test at α = .05 that μ = $260.

H_0: μ = $260
H_1: μ ≠ $260



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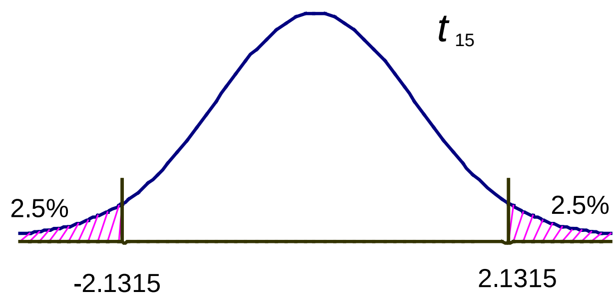


[NOTE: Z would be 1.96]

$t_{15} = \dfrac{200 - 260}{\frac{96}{\sqrt{16}}} = \dfrac{-60}{24} = -2.5$ $\quad p < .05 \quad$ REJECT H_0

(b) 95% CIE of μ:

$200 \pm 2.1315 \dfrac{96}{\sqrt{16}} =$

200 ± 51.16

$\$148.84 \leftrightarrow \$251.16$

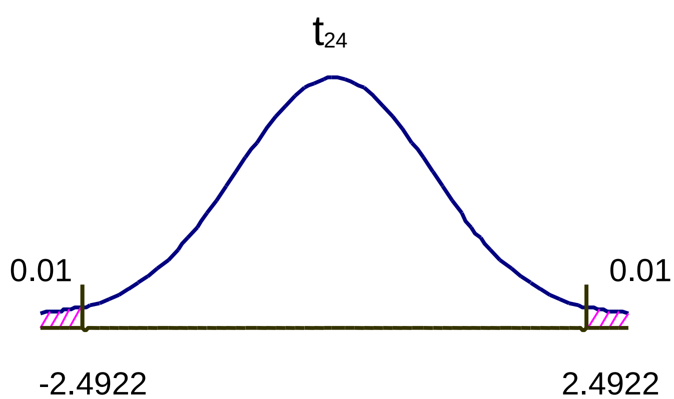
[Question: Is $260 covered by this interval estimator?]

EXAMPLE:
A company claims that its soup vending machines deliver (on average) exactly 4 ounces of soup. The company statistician finds that:

n =25
$\bar{X}$ =3.97 ounces
s =.04 ounces

(a) Test at $\alpha = .02$.
(b) Construct a 2-tailed 98% confidence interval using the sample evidence.

(a)
H_0: $\mu = 4$ oz
H_1: $\mu \neq 4$ oz



$$t_{24} = \frac{3.97 - 4}{.04/\sqrt{25}} = \frac{-.03}{.008} = -3.75$$ REJECT H_0

Therefore, reject H_0 at $P < .02$.

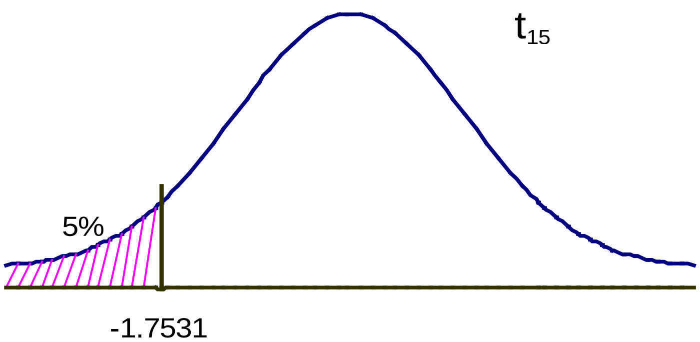
(b) $3.97 \pm 2.4922(.008) = 3.97 \pm .02$

3.95 oz ←———→ 3.99 oz

EXAMPLE:
A school claims that the average reading score of its students is at least 70.
n=16
$\bar{X}$ =68
s= 9

(a) Test at α = .05.
(b) Construct a 2-sided 95% confidence interval estimate of μ

(a) H_0: $\mu \geq 70$
H_1: $\mu < 70$

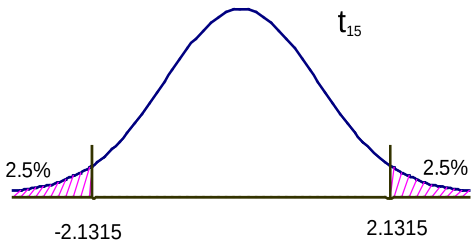


$$t_{15} = \frac{68 - 70}{9/\sqrt{16}} = \frac{-2}{2.25} = -0.88$$

DO NOT REJECT H_0

Therefore, do not reject H_0 at P > .05.

(b) 2-sided 95% Confidence Interval Estimate of μ:



$68 \pm 2.1315(2.25) = 68 \pm 4.8$

$63.2 \longleftarrow\!\!\!\longrightarrow 72.8$

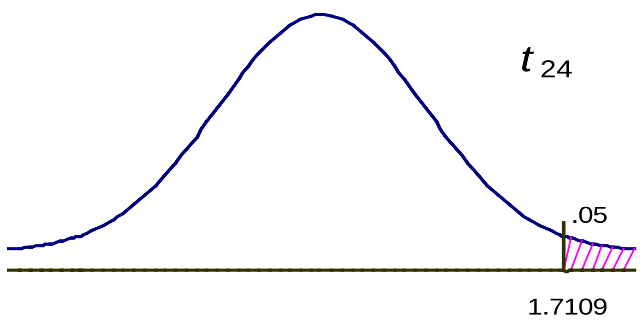
EXAMPLE:

The Perrier Company claims that at most there are 1 ppm (parts per million) of benzene in their water.

(a) Test at α=.05
(b) Construct a 2-tailed 95% C.I.E of μ

Data:
n=25
$\bar{X}$ = 1.16 ppm
s = .20 ppm

(a) H_0: $\mu \leq 1$ ppm
H_1: $\mu > 1$ ppm

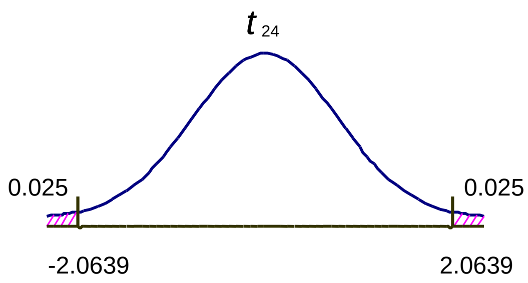


$$t_{24} = \frac{1.16 - 1.00}{.20/\sqrt{25}} = \frac{.16}{.04} = 4$$

REJECT H_0

Therefore, reject H_0 at $p < .05$.

(b) 2-sided 95% Confidence Interval Estimator of μ:



$1.16 \pm 2.0639(0.04)$

1.077 ppm ←——→ 1.243 ppm

EXAMPLE:

A company claims that cancer patients using drug X will live at least 10 more years.

$n=16$

$\bar{X} = 8.8$ year

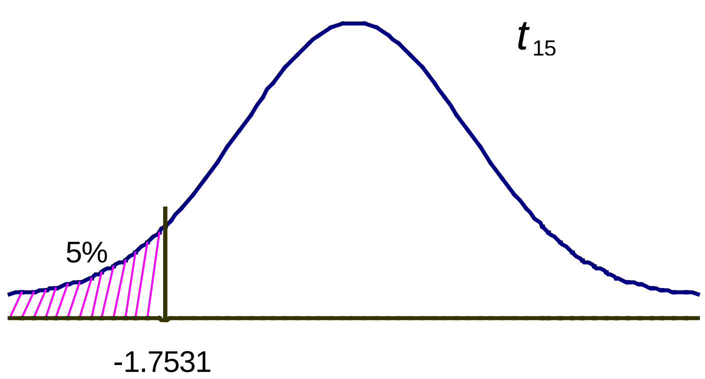
$s = 3.4$ year

(a) Test at $\alpha=.05$

(b) Construct a 2-sided 95% C.I.E

(a) H_0: $\mu \geq 10$ year

H_1: $\mu < 10$ year



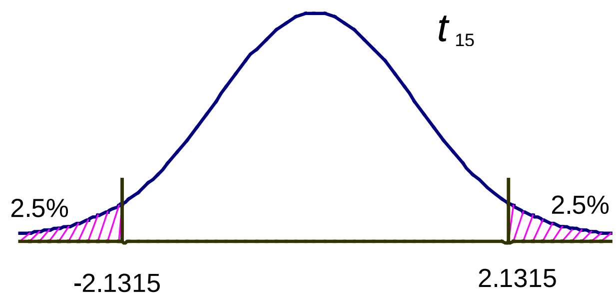
$$t_{15} = \frac{8.8-10}{3.4/\sqrt{16}} = \frac{-1.20}{.85} = -1.41$$

DO NOT REJECT H_0 $p > .05$

Therefore, do not reject H_0

.

(b) 2-sided 95% Confidence Interval



$$8.8 \pm 2.1315(0.85) \Rightarrow 8.8 \pm 1.81$$

6.99 years ⟵——⟶ 10.61 years

EXAMPLE:

A company claims that its batteries have a life of at least 550 hours. A firm receives a shipment and tests a sample of 25 batteries, and finds:

$\bar{X}$ = 530 hours

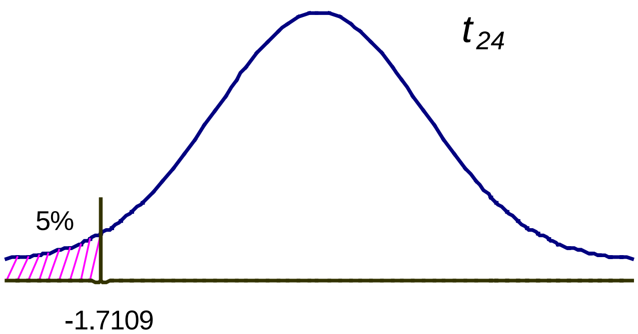
s = 37 hours

(a) Test at α=.05

(b) Construct a 2-tailed 95% C.I.E of μ

(a) H_0: $\mu \geq 550$ hours

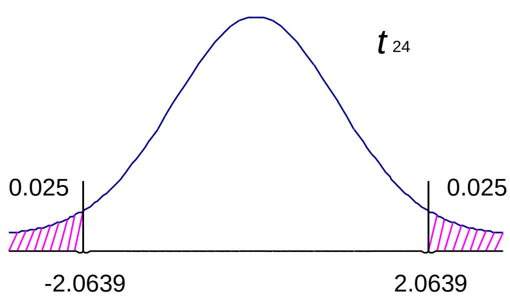
H_1: $\mu < 550$ hours



$$t_{24} = \frac{530 - 550}{37/\sqrt{25}} = \frac{-20}{7.4} = -2.7$$

REJECT H_0 at $p < .05$.

(b) 2-sided 95% C.I.E of μ



$$530 \pm 2.0639(7.4) \Rightarrow 530 \pm 15.27$$

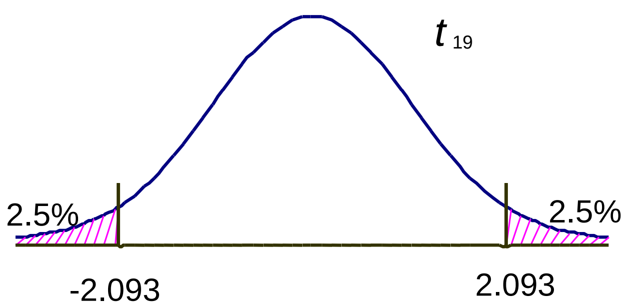
514.73 hours $\longleftrightarrow$ 545.27 hours

EXAMPLE:
A company manufactures bars of soap that are supposed to weigh exactly (on average) 10 ounces. A sample is taken:

n=20
$\overline{X}$ = 10.20 oz
s = 0.80 oz

(a) Test at $\alpha = .05$
(b) Construct a 95% CIE of μ

(a)
H_0: $\mu = 10$ oz
H_1: $\mu \neq 10$ oz



$$t_{19} = \frac{10.20 - 10.00}{\frac{.80}{\sqrt{20}}} = \frac{.20}{.18} = 1.11$$ DO NOT REJECT H_0

(b) 95% CIE of μ

$10.20 \pm 2.093\ (.18)$

$10.20 \pm .38$

$9.82 \leftrightarrow 10.58$ oz

EXAMPLE:

A company claims that its yogurt ice cream has no more than 2.0 mg of fat. Jerry and Elaine hire a researcher to conduct a statistical test of this claim. The researcher finds:

$n=25$

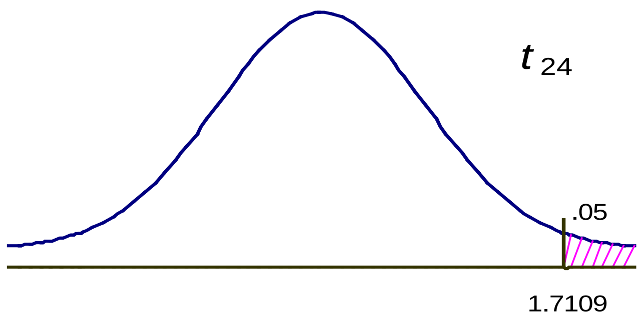
$\bar{X} = 2.2$ mg

$s = 0.3$ mg

Test the claim at $\alpha = .05$

H_0: $\mu \le 2$ mg

H_1: $\mu > 2$ mg



$t_{24} = \dfrac{2.2 - 2.0}{\frac{.3}{\sqrt{25}}} = \dfrac{.20}{.06} = 3.33$ REJECT H_0 $p < .05$

EXAMPLE: SUBSTANTIATING A CLAIM

XYZ Industries manufactures bottled water. According to government standards, there should not be more than 5 ppb (parts per billion) of lead in the product. The company needs to do a statistical test to substantiate this claim for a report to the relevant government agency.

They sampled 25 bottles and found:

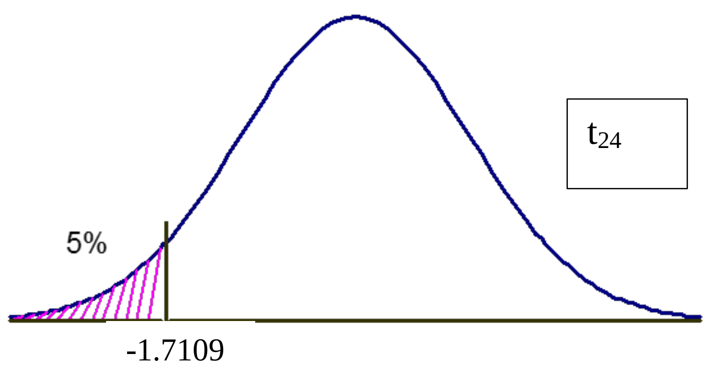
$\bar{X} = 4.2$ ppb

s = 0.4 ppb

Substantiate the claim at $\alpha = .05$

H_0: $\mu \geq 5$ ppb

H_1: $\mu < 5$ ppb



$$t_{24} = \frac{4.2 - 5.0}{\frac{.4}{\sqrt{25}}} = \frac{.80}{.08} = -10$$

REJECT H_0 $p < .05$

By rejecting H_0 the claim has been substantiated.

HYPOTHESIS TESTING – SUMMARY SO FAR:

We can perform various types of hypothesis tests, but the procedure is always the same. How do we know what type of test to perform? Here is what we have learned so far:

- Do we use Z or t?
- One-tail or two-tail test?

Still to come:

- Are we making hypotheses about μ or P (the population proportion)?
  [P is an optional topic]
- One sample or two samples? Are we looking at only one single set of data, or are we trying to compare two groups on the parameter of interest?